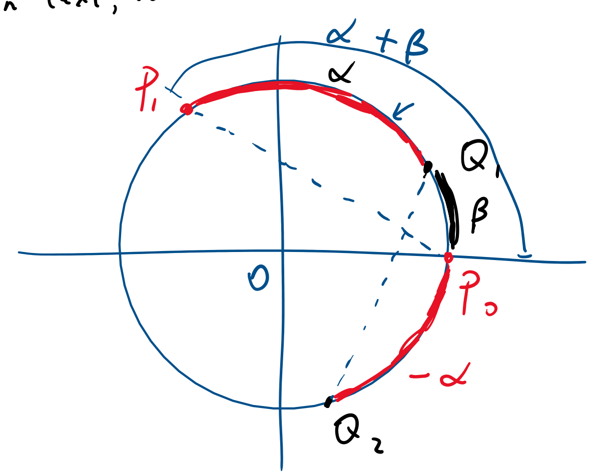
II. Add and Subtract

i. Difference of Cosine

$$\boxed{\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \end{aligned}}$$

Pf: ← In text, too



$d(P_0, P_1) = d(Q_0, Q_1)$ ← distance formula for arclength of α and β

Then, we have

$$\sqrt{(\cos(\alpha+\beta) - 1)^2 + (\sin(\alpha+\beta) - 0)^2} = \sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta + \sin\alpha)^2}$$

$$(\qquad)^2 \qquad (\qquad)^2$$

$$(\cos(\alpha+\beta) - 1)^2 + \sin^2(\alpha+\beta) = (\cos\beta - \cos\alpha)^2 + (\sin\beta + \sin\alpha)^2$$

$$\cos^2(\alpha+\beta) - 2\cos(\alpha+\beta) + 1 + \sin^2(\alpha+\beta) = \cos^2\beta - 2\cos\beta\cos\alpha + \cos^2\alpha + \sin^2\beta + 2\sin\beta\sin\alpha + \sin^2\alpha$$

$$1 - 2\cos(\alpha+\beta) + 1 = 1 \ldots$$

$\sin^2\beta + 2\sin\beta\sin\alpha + \sin^2\alpha$

$$1 - 2\cos(\alpha+\beta) + 1 = 1 - 2\cos\beta\cos\alpha + 2\sin\beta\sin\alpha + 1$$

$$\cancel{2} - 2\cos(\alpha+\beta) = \cancel{2} - 2\cos\beta\cos\alpha + 2\sin\beta\sin\alpha$$

$$-2\cos(\alpha+\beta) = -2\cos\beta\cos\alpha + 2\sin\beta\sin\alpha$$

$$\frac{\overset{1}{\cancel{-2}}\cos(\alpha+\beta)}{\underset{1}{\cancel{-2}}} = \overset{1}{\cancel{-2}}(\cos\beta\cos\alpha - \sin\beta\sin\alpha)$$

$$\cos(\alpha+\beta) = \cos\beta\cos\alpha - \sin\beta\sin\alpha$$

$$\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta} \checkmark$$

$5 \cdot y$

$y \cdot 5$

commute

$$\cos(\alpha-\beta) = \cos(\alpha + (-\beta))$$

$$= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$$

$$= \cos\alpha\cos\beta - \sin\alpha \cdot -\sin\beta$$

$$= \cos\alpha\cos\beta - -\sin\alpha\sin\beta$$

$$= \boxed{\cos\alpha\cos\beta + \sin\alpha\sin\beta} \checkmark$$

eg. Find the exact value of $\cos 15°$. ← exact value means non-decimal

Sol: $15°$ connects $\alpha+\beta$ or $\alpha-\beta$

$$15° = 60° - 45° = 45° - 30° = \cancel{0° + 15°} = \cancel{15° - 0°}$$

$$\cos 15° = \cos(45° - 30°)$$

$$= \cos 45°\cos 30° + \sin 45°\sin 30°$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$

$\sqrt{2}\cdot\sqrt{3}$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

eg. Find the exact value of $\cos 75°$.

Sol: $75° = \cancel{90° - 15°} = \underset{\text{No}}{120°} - \underset{\checkmark}{45°} = 30° + 45°$

$$\cos 75° = \cos(30° + 45°)$$

$$= \cos 30° \cos 45° - \sin 30° \sin 45°$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

eg. Find the exact value of $\cos\frac{\pi}{12}$.

Sol: $\frac{\pi}{12}$ is harder. We connect to $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ first

then turn each to $\frac{*}{12}$ ← $\uparrow\cdot\frac{2}{2}$ $\uparrow\cdot\frac{3}{3}$ $\uparrow\cdot\frac{4}{4}$ $\uparrow\cdot\frac{6}{6}$

$\frac{2\pi}{12}, \frac{3\pi}{12}, \frac{4\pi}{12}, \frac{6\pi}{12}$

$$\frac{\pi}{12} = \frac{3\pi}{12} - \frac{2\pi}{12}$$

$$= \frac{\pi}{4} - \frac{\pi}{6}$$

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$

$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$

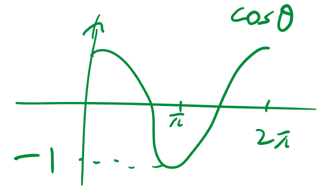
$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

$= \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$

$\frac{5\pi}{6} = \frac{?\pi}{6} + \frac{?\pi}{6}$

eg. Find the exact value of $\cos\frac{5\pi}{6}$.

Sol: $\frac{5\pi}{6} = \pi - \frac{\pi}{6} = \frac{\pi}{2} + \frac{\pi}{3}$



$\cos\pi = -1$

$\cos\frac{5\pi}{6} = \cos(\pi - \frac{\pi}{6})$

$= \cos\pi\cos\frac{\pi}{6} + \sin\pi\sin\frac{\pi}{6}$

$= -1\cdot\frac{\sqrt{3}}{2} + 0\cdot\frac{1}{2}$

$= \boxed{-\frac{\sqrt{3}}{2}}$