ii. Difference of Sine

$$\boxed{\begin{aligned} \sin(A+B) &= \sin A\cos B + \cos A\sin B \\ \sin(A-B) &= \sin A\cos B - \cos A\sin B \end{aligned}}$$

Pf: skip

eg. Find the exact value of $\sin\frac{\pi}{12}$.

sol:

$\frac{\pi}{12}$ connects to $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$

$\frac{2\pi}{12} \quad \frac{3\pi}{12} \quad \frac{4\pi}{12} \quad \frac{6\pi}{12} \quad \frac{12\pi}{12}$

$$\begin{aligned} \sin\frac{\pi}{12} &= \sin\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right) \\ &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} \\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\ &= \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}} \end{aligned}$$

$$\sqrt{3}\cdot\sqrt{2} = \sqrt{3\cdot 2} = \sqrt{6}$$

eg. Find the exact value of $\sin 105°$.

sol:

$$\begin{aligned} \sin 105° &= \sin(60° + 45°) \\ &= \sin 60°\cos 45° + \cos 60°\sin 45° \\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6}}{\cdot} + \frac{\sqrt{2}}{\cdot} \end{aligned}$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

eg. Find the exact value of $\sin\frac{7\pi}{12}$.

Sol: $\sin\frac{7\pi}{12} = \sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right)$

$\frac{7\pi}{12}$ connects to $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$

$\frac{2\pi}{12}$ $\frac{3\pi}{12}$ $\frac{4\pi}{12}$ $\frac{6\pi}{12}$ $\frac{12\pi}{12}$

$$= \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$

$$= \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$$

eg. Find the exact value of $\sin 15°$.

Sol: $\sin 15° = \sin(45° - 30°)$

$$= \sin 45°\cos 30° - \cos 45°\sin 30°$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

iii. Tangent

$$\boxed{\begin{aligned} \tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ \tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \end{aligned}}$$

" $\frac{A+B}{1-AB}$ "

" $\frac{A-B}{1+AB}$ "

Pf: $\tan(A+B) = \frac{\sin(A+B)}{\cos(A-B)}$

$= \ldots$

$\vdots$

$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$ ✓

eg. Find the exact value of $\tan \frac{7\pi}{12}$.

Sol: $\tan \frac{7\pi}{12} = \tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$ $\leftarrow$ $\frac{7\pi}{12} = \frac{3\pi}{12} + \frac{4\pi}{12}$ from above eg

$$= \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{3}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{3}}$$

$$= \frac{1+\sqrt{3}}{1 - 1\cdot\sqrt{3}}$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}} \cdot \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{(1+\sqrt{3})^2}{1^2-(\sqrt{3})^2}$$

$$= \frac{(1+\sqrt{3})^2}{1-3}$$

$$= \frac{(1+\sqrt{3})^2}{-2}$$

$$= \boxed{-\frac{(1+\sqrt{3})^2}{2}}$$

eg. Find the exact value of $\tan 15°$.

sol: $\tan 15° = \tan(60° - 45°)$ ← $60° - 45°$

$$= \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3} \cdot 1}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}} \cdot \frac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$= \frac{(\sqrt{3}-1) \cdot -(\sqrt{3}-1)}{1^2 - (\sqrt{3})^2}$$

$$= \frac{-(\sqrt{3}-1)^2}{1-3}$$

$$= \frac{-(\sqrt{3}-1)^2}{-2}$$

$a - b = -(b-a)$

eg. $x - 5 = -(5-x)$

$$= \frac{-(\sqrt{3}-1)^2}{-2}$$

$$= \boxed{\frac{(\sqrt{3}-1)^2}{2}}$$

eg. Verify the identity: $\tan\left(x - \frac{\pi}{4}\right) = \frac{\tan x - 1}{\tan x + 1}$.

Sol:

$$\tan\left(x - \frac{\pi}{4}\right) = \frac{\tan x - \tan\frac{\pi}{4}}{1 + \tan x \tan\frac{\pi}{4}}$$

$$= \frac{\tan x - 1}{1 + \tan x \cdot 1}$$

$$= \frac{\tan x - 1}{\tan x + 1} \checkmark$$

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eg $\sin(A+B) = \sin A\cos B + \cos A \sin B$

g. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

eg. $\cos(A - B) = \cos A\cos B + \sin A \sin B$