IV. Half-Angle Identity ← won't be tested ☺

be prepared for Calculus

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan\frac{A}{2} = \begin{cases} \dfrac{1-\cos A}{\sin A} \\ \dfrac{\sin A}{1+\cos A} \end{cases}$$

← be aware of ±, + or − is depending on the quadrants

eg. Evaluate $\cos 15^\circ$ ← 15°? Connect to 30°

Sol: $= \cos\frac{30^\circ}{2}$

$= \sqrt{\frac{1+\cos 30^\circ}{2}}$ ← 15° is at Q1, so we use $+\sqrt{\ }$

$= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$ ← ok!

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or $= \sqrt{\frac{\frac{2}{2}+\frac{\sqrt{3}}{2}}{2}}$

$= \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}$ ← $\frac{2+\sqrt{3}}{2} \cdot \frac{1}{2}$

$= \sqrt{\frac{2+\sqrt{3}}{4}}$

$$= \frac{\sqrt{2+\sqrt{3}}}{\sqrt{4}}$$

$$= \boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}$$

eg. Find the exact value of $\sin(-15°)$

$$\boxed{\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}}$$

Sol: $= \sin\frac{-30°}{2}$

$= -\sqrt{\frac{1-\cos 30°}{2}}$ ← $-15°$ is in Q4, so it's $-\sqrt{\ }$

$= \boxed{-\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}}$ ← ok

$= -\sqrt{\frac{\frac{2}{2}-\frac{\sqrt{3}}{2}}{2}}$

$= -\sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}$ ← $\frac{2-\sqrt{3}}{2} \cdot \frac{1}{2}$

$= -\sqrt{\frac{2-\sqrt{3}}{4}}$

$= -\frac{\sqrt{2-\sqrt{3}}}{\sqrt{4}}$

$= \boxed{-\frac{\sqrt{2-\sqrt{3}}}{2}}$

Eg.

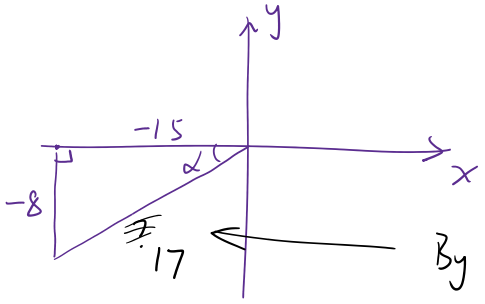
Finding Exact Values Using Half-Angle Identities

Given that $\tan\alpha = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

ⓐ $\sin\left(\frac{\alpha}{2}\right)$ ⓑ $\cos\left(\frac{\alpha}{2}\right)$ ⓒ $\tan\left(\frac{\alpha}{2}\right)$

Sol:

$\tan\alpha = \frac{8}{15}$, α in QIII ← very easy!



By Pythagorean Theorem:

$(-8)^2 + (-15)^2 = r^2$

$64 + 225 = r^2$

$\sqrt{289} = \sqrt{r^2}$

$\pm 17 = r$

$17 = r$ ← because radius

ⓐ $\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$

$= \sqrt{\frac{1-\left(-\frac{15}{17}\right)}{2}}$

$= \sqrt{\frac{1+\frac{15}{17}}{2}}$

$= \sqrt{\frac{\frac{17}{17}+\frac{15}{17}}{2}}$

$= \sqrt{\frac{\frac{32}{17}}{2}}$

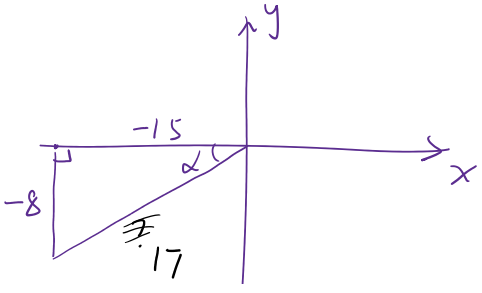
$= \sqrt{\frac{16}{17}}$

$= \frac{4}{\sqrt{17}} \cdot \frac{\sqrt{17}}{\sqrt{17}}$

$= \boxed{\frac{4\sqrt{17}}{17}}$

← α is in QIII, eg. $\alpha = 220°$, $\frac{\alpha}{2}$ is in QII 110°

ⓑ $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$



(b) $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

$= -\sqrt{\frac{1+(-\frac{15}{17})}{2}}$

$= -\sqrt{\frac{1-\frac{15}{17}}{2}}$

$= -\sqrt{\frac{\frac{17}{17}-\frac{15}{17}}{2}}$

$= -\sqrt{\frac{\frac{2}{17}}{2}}$

$= -\sqrt{\frac{1}{17}}$

$= -\frac{\sqrt{1}}{\sqrt{17}}$

$= -\frac{1}{\sqrt{17}}\cdot\frac{\sqrt{17}}{\sqrt{17}}$

$= \boxed{-\frac{\sqrt{17}}{17}}$

α is in QIII, $\frac{\alpha}{2}$ is in QII

eg. 220° → 110°

$\frac{8}{17}\cdot\frac{1}{2}$

(c) $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$

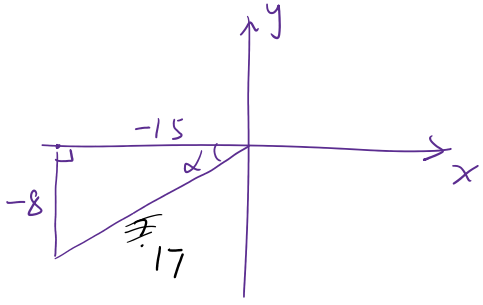
$= \frac{-\frac{8}{17}}{1+\frac{-15}{17}}$

$= \frac{-\frac{8}{17}}{1-\frac{15}{17}}$

$= \frac{-\frac{8}{17}}{\frac{17}{17}-\frac{15}{17}}$

$= \frac{-\frac{8}{17}}{\frac{2}{17}}$

$= -\frac{8}{2}\times\frac{17}{1}$



$$\tan\frac{A}{2} = \begin{cases} \frac{1-\cos A}{\sin A} \\ \frac{\sin A}{1+\cos A} \end{cases}$$

$$= -\frac{\cancel{8}^{4}}{\cancel{17}_{1}} \cdot \frac{\cancel{17}^{1}}{\cancel{8}_{1}}$$

$$= \boxed{-4}$$